IV. *On Hamilton's Numbers.*—Part II.

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§ 4. *Continuation, to an infinite number of terms, of the Asymptotic Development for Hypothenusal Numbers.*

"This was sometime a paradox, but now the time gives it proof."
(*Hamlet*, Act III., scene 1.)

IN the third section of this paper ('Phil. Trans.,' A., vol. 178, p. 311) it was stated, on what is now seen to be insufficient evidence, that the asymptotic development of $p - q$, the half of any Hypothenusal Number, could be expressed as a series of powers of $q - r$, the half of its antecedent, in which the indices followed the sequence

$$2, \tfrac{3}{2}, 1, \tfrac{3}{4}, \tfrac{5}{8}, \tfrac{1}{2}, \ldots$$

It was there shown that, when quantities of an order of magnitude inferior to that of $(q - r)^{\frac{3}{4}}$ are neglected,

$$p - q = (q - r)^2 + \tfrac{4}{3}(q - r)^{\frac{3}{2}} + \tfrac{11}{18}(q - r) + \tfrac{10}{81}(q - r)^{\frac{3}{4}};$$

but, on attempting to carry this development further, it was found that, though the next term came out $\frac{88}{1215}(q - r)^{\frac{5}{8}}$, there was an infinite series of terms interposed between this one and $(q - r)^{\frac{1}{2}}$, viz., as proved in the present section, between $(q - r)^{\frac{3}{4}}$ and $(q - r)^{\frac{1}{2}}$ there lies an infinite series of terms whose indices are

$$\tfrac{5}{8}, \tfrac{9}{16}, \tfrac{17}{32}, \tfrac{33}{64}, \tfrac{65}{128}, \ldots,$$

and whose coefficients form a geometrical series of which the first term is $\frac{88}{1215}$ and the common ratio $\frac{2}{3}$.

We shall assume the law of the indices (which, it may be remarked, is identical with that given in the introduction to this paper as originally printed in the 'Proceedings,' but subsequently altered in the 'Transactions') and write

$$
\begin{aligned}
p - q = {} & (q-r)^2 + \tfrac{4}{3}(q-r)^{\frac{3}{2}} + \tfrac{11}{18}(q-r) + \tfrac{10}{81}(q-r)^{\frac{3}{4}} \\
& + \tfrac{2^3}{3^3}\,\mathrm{A}\,(q-r)^{\frac{5}{8}} + \tfrac{2^4}{3^4}\,\mathrm{B}\,(q-r)^{\frac{9}{16}} + \tfrac{2^5}{3^5}\,\mathrm{C}\,(q-r)^{\frac{17}{32}} \\
& + \tfrac{2^6}{3^6}\,\mathrm{D}\,(q-r)^{\frac{33}{64}} + \tfrac{2^7}{3^7}\,\mathrm{E}\,(q-r)^{\frac{65}{128}} + \text{\&c., ad inf.} \\
& + \Theta\,^{*} \quad \ldots \ldots \ldots \qquad (1)
\end{aligned}
$$

The law of the coefficients will then be established by proving that

$$\mathrm{A} = \mathrm{B} = \mathrm{C} = \mathrm{D} = \mathrm{E} = \ldots = \tfrac{11}{45}.$$

If there were any terms, of an order superior to that of $(q-r)^{\frac{1}{2}}$, whose indices did not obey the assumed law, any such term would make its presence felt in the course of the work; for, in the process we shall employ, the coefficient of each term has to be determined before that of any subsequent term can be found. It was in this way that the existence of terms between $(q-r)^{\frac{5}{8}}$ and $(q-r)^{\frac{1}{2}}$ was made manifest in the unsuccessful attempt to calculate the coefficient of $(q-r)^{\frac{1}{2}}$. It thus appears that the assumed law of the indices is the true one.

It will be remembered that $p, q, r, \ldots,$ are the halves of the sharpened Hamiltonian Numbers $\mathrm{E}_{n+1}, \mathrm{E}_n, \mathrm{E}_{n-1}, \ldots,$ and that consequently the relation

$$\mathrm{E}_{n+1} = 1 + \frac{\mathrm{E}_n(\mathrm{E}_n - 1)}{1.2} - \frac{\mathrm{E}_{n-1}(\mathrm{E}_{n-1}-1)(\mathrm{E}_{n-1}-2)}{1.2.3} + \ldots$$

may be written in the form

$$
\begin{aligned}
p = {} & \tfrac{1}{2} + \frac{q(2q-1)}{2} - \frac{r(2r-1)(2r-2)}{2.3} + \frac{s(2s-1)(2s-2)(2s-3)}{2.3.4} \\
& - \frac{t(2t-1)(2t-2)(2t-3)(2t-4)}{2.3.4.5} + \frac{u(2u-1)(2u-2)(2u-3)(2u-4)(2u-5)}{2.3.4.5.6} \\
& - \ldots \quad \ldots \ldots \ldots \qquad (2)
\end{aligned}
$$

The comparison of this value of p with that given by (1) furnishes an equation which, after several reductions have been made, in which special attention must be paid to the order of the quantities under consideration, ultimately leads to the determination of the values of A, B, C, . . . , in succession.

Taking unity to represent the order of q, the orders of

$$p, \; q, \; r, \; s, \; t, \; u, \; v, \; w, \; \ldots$$

will be

$$2, \; 1, \; \tfrac{1}{2}, \; \tfrac{1}{4}, \; \tfrac{1}{8}, \; \tfrac{1}{16}, \; \tfrac{1}{32}, \; \tfrac{1}{64}, \; \ldots$$

Hence, after expanding each of the binomials on the right-hand side of (1) and arranging the terms in descending order, retaining only terms for which the order is superior to $\frac{1}{2}$, we shall find

* In the text above Θ represents some unknown function, the asymptotic value of whose ratio to $(q-r)^{\frac{1}{2}}$ is not infinite.

$$
\begin{array}{lll}
\text{Order} & 2 & p = q^2 \\
\quad ,, & \frac{3}{2} & \quad - 2qr + \frac{4}{3} q^{\frac{3}{2}} \\
\quad ,, & 1 & \quad + r^2 - 2q^{\frac{1}{2}}r + \frac{29}{18} q \\
\quad ,, & \frac{3}{4} & \qquad\qquad + \frac{10}{81} q^{\frac{3}{4}} \\
\quad ,, & \frac{5}{8} & \qquad\qquad + \frac{2^3}{3^3} A\, q^{\frac{5}{8}} \\
\quad ,, & \frac{9}{16} & \qquad\qquad + \frac{2^4}{3^4} B\, q^{\frac{9}{16}} \\
\quad ,, & \frac{17}{32} & \qquad\qquad + \frac{2^5}{3^5} C\, q^{\frac{17}{32}} \\
\quad ,, & \frac{33}{64} & \qquad\qquad + \frac{2^6}{3^6} D\, q^{\frac{33}{64}} \\
\quad ,, & \frac{65}{128} & \qquad\qquad + \frac{2^7}{3^7} E\, q^{\frac{65}{128}} + \ldots\ldots \qquad (3)
\end{array}
$$

Again, retaining only those terms of (2) whose order is superior to $\frac{1}{2}$, we have

$$
\begin{array}{lccccc}
 & p = q^2; & -\frac{2}{3} r^3; & -\frac{1}{2} q + r^2 + \frac{1}{3} s^4; & -s^3; & -\frac{2}{15} t^5 \quad \ldots\ldots \quad (4) \\
\text{Order} & 2; & \frac{3}{2}; & 1; & \frac{3}{4}; & \frac{5}{8}.
\end{array}
$$

From (3) and (4) we obtain by subtraction

$$
\begin{array}{lll}
\text{Order} & \frac{3}{2} & 0 = \frac{2}{3} r^3 - 2qr + \frac{4}{3} q^{\frac{3}{2}} \\
\quad ,, & 1 & \quad - \frac{1}{3} s^4 - 2q^{\frac{1}{2}}r + \frac{19}{9} q \\
\quad ,, & \frac{3}{4} & \qquad + s^3 + \frac{10}{81} q^{\frac{3}{4}} \\
\quad ,, & \frac{5}{8} & \qquad + \frac{2}{15} t^5 + \frac{2^3}{3^3} A\, q^{\frac{5}{8}} \\
\quad ,, & \frac{9}{16} & \qquad\qquad + \frac{2^4}{3^4} B\, q^{\frac{9}{16}} \\
\quad ,, & \frac{17}{32} & \qquad\qquad + \frac{2^5}{3^5} C\, q^{\frac{17}{32}} \\
\quad ,, & \frac{33}{64} & \qquad\qquad + \frac{2^6}{3^6} D\, q^{\frac{33}{64}} \\
\quad ,, & \frac{65}{128} & \qquad\qquad + \frac{2^7}{3^7} E\, q^{\frac{65}{128}} + \ldots\ldots \qquad (5)
\end{array}
$$

Changing $p, q, r, \ldots$ into $q, r, s, \ldots$ respectively, equation (4) becomes

$$q = r^2 - \frac{2}{3} s^3 - \frac{1}{2} r + s^2 + \frac{1}{3} t^4 - t^3 - \frac{2}{15} u^5,$$

so that, if we assume $q = r^2(1 - \alpha)$, the order of α will be the same as that of $r^{-2}s^3$, viz., $-\frac{2}{2} + \frac{3}{4} = -\frac{1}{4}$.

Hence, if we substitute $r^2(1 - \alpha)$ for q in (5), neglecting in the result quantities of the order $\frac{1}{2}$, we shall find

$$
\begin{aligned}
&\frac{2}{3} r^3 - 2qr + \frac{4}{3} q^{\frac{3}{2}} - \frac{1}{3} s^4 - 2q^{\frac{1}{2}}r + \frac{19}{9} q \\
&= \frac{2}{3} r^3 - 2r^3(1 - \alpha) + \frac{4}{3} r^3 (1 - \frac{3}{2}\alpha + \frac{3}{8}\alpha^2 + \frac{1}{16}\alpha^3) \\
&\quad - \frac{1}{3} s^4 - 2r^2(1 - \frac{1}{2}\alpha) + \frac{19}{9} r^2 (1 - \alpha) \\
&= \frac{1}{2} r^3\alpha^2 + \frac{1}{12} r^3\alpha^3 - \frac{1}{3} s^4 + \frac{1}{9} r^2 - \frac{10}{9} r^2\alpha;
\end{aligned}
$$

while at the same time, since the order of $r^{\frac{3}{2}}\alpha$ does not exceed $\frac{1}{2}$, we have

$$q^{\frac{3}{4}} = r^{\frac{3}{2}}(1-\alpha)^{\frac{3}{2}} = r^{\frac{3}{2}},$$

and in like manner

$$q^{\frac{5}{8}} = r^{\frac{5}{4}},\ q^{\frac{9}{16}} = r^{\frac{9}{8}}, \text{ and so on.}$$

Thus equation (5) becomes

$$\begin{array}{lll}
\text{Order} & 1 & 0 = \frac{1}{2}r^3\alpha^2 - \frac{1}{3}s^4 + \frac{1}{9}r^2 \\
\text{,,} & \frac{3}{4} & \quad + \frac{1}{12}r^3\alpha^3 - \frac{10}{9}r^2\alpha + s^3 + \frac{10}{81}r^{\frac{3}{2}} \\
\text{,,} & \frac{5}{8} & \quad + \frac{2}{15}t^5 + \frac{2^3}{3^3}\text{A}\, r^{\frac{5}{4}} \\
\text{,,} & \frac{9}{16} & \quad + \frac{2^4}{3^4}\text{B}\, r^{\frac{9}{8}} \\
\text{,,} & \frac{17}{32} & \quad + \frac{2^5}{3^5}\text{C}\, r^{\frac{17}{16}} \\
\text{,,} & \frac{33}{64} & \quad + \frac{2^6}{3^6}\text{D}\, r^{\frac{33}{32}} \\
\text{,,} & \frac{65}{128} & \quad + \frac{2^7}{3^7}\text{E}\, r^{\frac{65}{64}} + \dots
\end{array} \qquad (6)$$

where

$$\alpha = \frac{2}{3}r^{-2}s^3 ;\ + \frac{1}{2}r^{-1} - r^{-2}s^2 - \frac{1}{3}r^{-2}t^4 ;\ + r^{-2}t^3 ;\ + \frac{2}{15}r^{-2}u^5$$

order

$$-\frac{1}{4} \quad ; \quad -\frac{1}{2} \quad ; \quad -\frac{5}{8} \quad ; \quad -\frac{11}{16}$$

Let

$$\alpha = \frac{2}{3}r^{-2}s^3(1+\alpha')$$

then

$$\alpha' = \frac{3}{2}s^{-3}\left(\frac{1}{2}r - s^2 - \frac{1}{3}t^4 + t^3 + \frac{2}{15}u^5\right)$$

where terms as far as, but not beyond, $-\frac{7}{16}$ (which is the order of $s^{-3}u^5$) have been retained.

Now

p	consists of terms whose orders are	2,	$\frac{3}{2}$,	1,	$\frac{3}{4}$,	$\frac{5}{8}$,	$\frac{1}{2}, \dots$
q	,, ,, ,,	1,	$\frac{3}{4}$,	$\frac{1}{2}$,	$\frac{3}{8}$,	$\frac{5}{16}$,	$\frac{1}{4}, \dots$
α	,, ,, ,,		$-\frac{1}{4}$,	$-\frac{1}{2}$,	$-\frac{5}{8}$,	$-\frac{11}{16}$,	$-\frac{3}{4}, \dots$
α'	,, ,, ,,			$-\frac{1}{4}$,	$-\frac{3}{8}$,	$-\frac{7}{16}$,	$-\frac{1}{2}, \dots$

Thus the order of α' is $-\frac{1}{4}$, and in the above expression all terms of α' superior to $-\frac{1}{2}$ have been retained, and consequently (rejecting the square of α' whose order is $-\frac{1}{2}$) in the first line of (6) we may write

$$\begin{aligned}
\tfrac{1}{2}r^3\alpha^2 &= \tfrac{2}{9}r^{-1}s^6(1+2\alpha') \\
&= \tfrac{2}{9}r^{-1}s^6 + \tfrac{2}{3}r^{-1}s^3\left(\tfrac{1}{2}r - s^2 - \tfrac{1}{3}t^4 + t^3 + \tfrac{2}{15}u^5\right) \\
&= \tfrac{2}{9}r^{-1}s^6 + \tfrac{1}{3}s^3 - \tfrac{2}{3}r^{-1}s^5 - \tfrac{2}{9}r^{-1}s^3t^4 + \tfrac{2}{3}r^{-1}s^3t^3 + \tfrac{4}{45}r^{-1}s^3u^5.
\end{aligned}$$

In the second line of (6) we may reject the whole of α', since its order is $-\frac{1}{4}$, and write

$$\begin{aligned}&\tfrac{1}{12}r^3\alpha^3 - \tfrac{10}{9}r^2\alpha + s^3\\ &= \tfrac{2}{81}r^{-3}s^9 + \tfrac{7}{27}s^3.\end{aligned}$$

After substituting their values for the terms in (6) which contain α, and at the same time dividing throughout by $\frac{2}{3}$, we shall obtain

$$\begin{array}{lll}
\text{Order} & 1 & 0 = \tfrac{1}{3}r^{-1}s^6 - \tfrac{1}{2}s^4 + \tfrac{1}{6}r^2 \\
\quad ,, & \tfrac{3}{4} & \quad + \tfrac{1}{27}r^{-3}s^9 - r^{-1}s^5 - \tfrac{1}{3}r^{-1}s^3t^4 + \tfrac{8}{9}s^3 + \tfrac{5}{27}r^{\frac{3}{2}} \\
\quad ,, & \tfrac{5}{8} & \quad + r^{-1}s^3t^3 + \tfrac{1}{5}t^5 + \tfrac{2^2}{3^2}Ar^{\frac{5}{4}} \\
\quad ,, & \tfrac{9}{16} & \qquad + \tfrac{2}{15}r^{-1}s^3u^5 + \tfrac{2^3}{3^3}B\,r^{\frac{9}{8}} \\
\quad ,, & \tfrac{17}{32} & \qquad\qquad + \tfrac{2^4}{3^4}C\,r^{\frac{17}{16}} \\
\quad ,, & \tfrac{33}{64} & \qquad\qquad + \tfrac{2^5}{3^5}D\,r^{\frac{33}{32}} \\
\quad ,, & \tfrac{65}{128} & \qquad\qquad + \tfrac{2^6}{3^6}E\,r^{\frac{65}{64}} + \ldots \qquad \ldots \qquad (7)
\end{array}$$

We now write

$$r = s^2(1-\beta) \quad \text{and} \quad \beta = \tfrac{2}{3}s^{-2}t^3(1+\beta')$$

where, observing that the values of β and β' can be immediately deduced from those of α and α' by changing $r, s, t, \ldots$ into $s, t, u, \ldots$, it is evident that β and β' are both of the order $-\frac{1}{8}$; for α and α' are both of the order $-\frac{1}{4}$. Thus (neglecting quantities whose order is equal to, or less than, $\frac{1}{2}$) we have

$$\begin{aligned}
&\tfrac{1}{3}r^{-1}s^6 - \tfrac{1}{2}s^4 + \tfrac{1}{6}r^2\\
&= \tfrac{1}{3}s^4(1+\beta+\beta^2+\beta^3) - \tfrac{1}{2}s^4 + \tfrac{1}{6}s^4(1-2\beta+\beta^2) = \tfrac{1}{2}s^4\beta^2 + \tfrac{1}{3}s^4\beta^3\\
&= \tfrac{2}{9}t^6(1+2\beta') + \tfrac{8}{81}s^{-2}t^9\\
&= \tfrac{2}{9}t^6 + \tfrac{2}{3}t^3(\tfrac{1}{2}s - t^2 - \tfrac{1}{3}u^4 + u^3 + \tfrac{2}{15}v^5) + \tfrac{8}{81}s^{-2}t^9\\
&= \tfrac{2}{9}t^6\,; \ + \tfrac{1}{3}st^3 - \tfrac{2}{3}t^5 - \tfrac{2}{9}t^3u^4 + \tfrac{8}{81}s^{-2}t^9\,; \ + \tfrac{2}{3}t^3u^3\,; \ + \tfrac{4}{45}t^3v^5.
\end{aligned}$$

Order $\quad \frac{3}{4}\ ; \qquad \frac{5}{8} \qquad ; \quad \frac{9}{16} \quad ; \quad \frac{17}{32}$.

$$\begin{aligned}
&\tfrac{1}{27}r^{-3}s^9 - r^{-1}s^5 - \tfrac{1}{3}r^{-1}s^3t^4 + \tfrac{8}{9}s^3 + \tfrac{5}{27}r^{\frac{3}{2}}\\
&= \tfrac{1}{27}s^3(1+3\beta) - s^3(1+\beta) - \tfrac{1}{3}st^4(1+\beta) + \tfrac{8}{9}s^3 + \tfrac{5}{27}s^3(1-\tfrac{3}{2}\beta)\\
&= \tfrac{1}{9}s^3 - \tfrac{7}{6}s^3\beta - \tfrac{1}{3}st^4(1+\beta)\\
&= \tfrac{1}{9}s^3 - \tfrac{1}{3}st^4\,; \ - \tfrac{7}{9}st^3 - \tfrac{2}{9}s^{-1}t^7.
\end{aligned}$$

Order $\quad \frac{3}{4} \quad ; \qquad \frac{5}{8}$.

$$r^{-1}s^3t^3 + \tfrac{1}{5}t^5 + \tfrac{2^2}{3^2}\mathrm{A}r^{\frac{5}{4}} = st^3 + \tfrac{1}{5}t^5 + \tfrac{2^2}{3^2}\mathrm{A}s^{\frac{5}{2}},$$
$$\tfrac{2}{15}r^{-1}s^3u^5 + \tfrac{2^3}{3^3}\mathrm{B}r^{\frac{9}{8}} = \tfrac{2}{15}su^5 + \tfrac{2^3}{3^3}\mathrm{B}s^{\frac{9}{4}},$$

and so on.

Hence (7) becomes

$$\begin{array}{lll}
\text{Order} & \tfrac{3}{4} & 0 = \tfrac{2}{9}t^6 - \tfrac{1}{3}st^4 + \tfrac{1}{9}s^3 \\
\quad ,, & \tfrac{5}{8} & \quad + \tfrac{8}{81}s^{-2}t^9 - \tfrac{7}{15}t^5 - \tfrac{2}{9}t^3u^4 + \tfrac{5}{9}st^3 - \tfrac{2}{9}s^{-1}t^7 + \tfrac{2^2}{3^2}\mathrm{A}s^{\frac{5}{2}} \\
\quad ,, & \tfrac{9}{16} & \quad + \tfrac{2}{3}t^3u^3 + \tfrac{2}{15}su^5 + \tfrac{2^3}{3^3}\mathrm{B}s^{\frac{9}{4}} \\
\quad ,, & \tfrac{17}{32} & \quad + \tfrac{4}{45}t^3v^5 + \tfrac{2^4}{3^4}\mathrm{C}s^{\frac{17}{8}} \\
\quad ,, & \tfrac{33}{64} & \quad + \tfrac{2^5}{3^5}\mathrm{D}s^{\frac{33}{16}} \\
\quad ,, & \tfrac{65}{128} & \quad + \tfrac{2^6}{3^6}\mathrm{E}s^{\frac{65}{32}} + \ldots\ldots\ldots \quad (8)
\end{array}$$

Dividing this throughout by $\frac{2}{3}s$, and then writing

$$s = t^2(1-\gamma) \quad \text{and} \quad \gamma = \tfrac{2}{3}t^{-2}u^3(1+\gamma'),$$

we obtain in exactly the same manner as before, merely altering the letters in the previous work,

$$\begin{array}{l}
\tfrac{1}{3}s^{-1}t^6 - \tfrac{1}{2}t^4 + \tfrac{1}{6}s^2 \\
= \tfrac{2}{9}u^6\,;\ + \tfrac{1}{3}tu^3 - \tfrac{2}{3}u^5 - \tfrac{2}{9}u^3v^4 + \tfrac{8}{81}t^{-2}u^9\,;\ + \tfrac{2}{3}u^3v^3\,;\ + \tfrac{4}{45}u^3w^5.
\end{array}$$

Order $\quad \frac{3}{8}\ ; \quad \frac{5}{16} \quad ; \quad \frac{9}{32} \quad ; \quad \frac{17}{64}$

where quantities of the order $\frac{1}{4}$, or less, are now neglected.

Similarly

$$\begin{aligned}
&\tfrac{4}{27}s^{-3}t^9 - \tfrac{7}{10}s^{-1}t^5 - \tfrac{1}{3}s^{-1}t^3u^4 + \tfrac{5}{6}t^3 - \tfrac{1}{3}s^{-2}t^7 + \tfrac{2}{3}\mathrm{A}s^{\frac{3}{2}} \\
&= \tfrac{4}{27}t^3(1+3\gamma) - \tfrac{7}{10}t^3(1+\gamma) - \tfrac{1}{3}tu^4(1+\gamma) + \tfrac{5}{6}t^3 - \tfrac{1}{3}t^3(1+2\gamma) \\
&\qquad\qquad + \tfrac{2}{3}\mathrm{A}t^3(1 - \tfrac{3}{2}\gamma) \\
&= (\tfrac{2}{3}\mathrm{A} - \tfrac{7}{135})t^3 - \tfrac{1}{3}tu^4 - (\mathrm{A} + \tfrac{83}{90})t^3\gamma - \tfrac{1}{3}tu^4\gamma \\
&= (\tfrac{2}{3}\mathrm{A} - \tfrac{7}{135})t^3 - \tfrac{1}{3}tu^4\,;\ - (\tfrac{2}{3}\mathrm{A} + \tfrac{83}{135})tu^3 - \tfrac{2}{9}t^{-1}u^7.
\end{aligned}$$

Order $\quad \frac{3}{8} \quad ; \quad \frac{5}{16}$

$$s^{-1}t^3u^3 + \tfrac{1}{5}u^5 + \tfrac{2^2}{3^2}\mathrm{B}s^{\frac{5}{4}} = tu^3 + \tfrac{1}{5}u^5 + \tfrac{2^2}{3^2}\mathrm{B}t^{\frac{5}{2}}$$
$$\tfrac{2}{15}s^{-1}t^3v^5 + \tfrac{2^3}{3^3}\mathrm{C}s^{\frac{9}{8}} = \tfrac{2}{15}tv^5 + \tfrac{2^3}{3^3}\mathrm{C}t^{\frac{9}{4}},$$

and so on.

Thus (8) becomes

$$
\begin{array}{lll}
\text{Order } \frac{3}{8} & 0 = \frac{2}{9}u^6 - \frac{1}{3}tu^4 + \left(\frac{2}{3}\mathrm{A} - \frac{7}{135}\right)t^3 \\
\quad ,, \quad \frac{5}{16} & \quad + \frac{8}{81}t^{-2}u^9 - \frac{7}{15}u^5 - \frac{2}{9}u^3v^4 + \left(\frac{97}{135} - \frac{2}{3}\mathrm{A}\right)tu^3 - \frac{2}{9}t^{-1}u^7 + \frac{2^2}{3^2}\mathrm{B}t^{\frac{5}{2}} \\
\quad ,, \quad \frac{9}{32} & \quad + \frac{2}{3}u^3v^3 + \frac{2}{15}tv^5 + \frac{2^3}{3^3}\mathrm{C}t^{\frac{9}{4}} \\
\quad ,, \quad \frac{17}{64} & \qquad\qquad + \frac{4}{45}u^3w^5 + \frac{2^4}{3^4}\mathrm{D}t^{\frac{17}{8}} \\
\quad ,, \quad \frac{33}{128} & \qquad\qquad\qquad\qquad + \frac{2^5}{3^5}\mathrm{E}t^{\frac{33}{16}} + \ldots
\end{array}
$$

Now the terms of the highest order in this equation must vanish when we write $t = u^2$, and therefore $\frac{2}{9} - \frac{1}{3} + \frac{2}{3}\mathrm{A} - \frac{7}{135} = 0$, which gives $\mathrm{A} = \frac{11}{45}$. Substituting this value for A, we find

$$
\begin{array}{lll}
\text{Order } \frac{3}{8} & 0 = \frac{2}{9}u^6 - \frac{1}{3}tu^4 + \frac{1}{9}t^3 \\
\quad ,, \quad \frac{5}{16} & \quad + \frac{8}{81}t^{-2}u^9 - \frac{7}{15}u^5 - \frac{2}{9}u^3v^4 + \frac{5}{9}tu^3 - \frac{2}{9}t^{-1}u^7 + \frac{2^2}{3^2}\mathrm{B}t^{\frac{5}{2}} \\
\quad ,, \quad \frac{9}{32} & \quad + \frac{2}{3}u^3v^3 + \frac{2}{15}tv^5 + \frac{2^3}{3^3}\mathrm{C}t^{\frac{9}{4}} \\
\quad ,, \quad \frac{17}{64} & \qquad\qquad + \frac{4}{45}u^3w^5 + \frac{2^4}{3^4}\mathrm{D}t^{\frac{17}{8}} \\
\quad ,, \quad \frac{33}{128} & \qquad\qquad\qquad\qquad + \frac{2^5}{3^5}\mathrm{E}t^{\frac{33}{16}} + \ldots,
\end{array}
$$

which is a mere repetition of equation (8), with all the letters moved forward one place. Hence it is evident that, if we treat this equation as we treated (8), we shall find $\mathrm{B} = \frac{11}{45}$, arriving, at the same time, at another equation which will be merely a repetition of (8), with all its letters moved forward two places; and this process can be continued as long as we please.

Thus we arrive at the result—

$$\mathrm{A} = \mathrm{B} = \mathrm{C} = \mathrm{D} = \mathrm{E} = \ldots = \tfrac{11}{45},$$

and the asymptotic development for Hypothenusal Numbers

$$
\begin{aligned}
p - q = (q - r)^2 &+ \tfrac{4}{3}(q - r)^{\frac{3}{2}} + \tfrac{11}{18}(q - r) + \tfrac{10}{81}(q - r)^{\frac{3}{4}} \\
&+ \tfrac{11}{45}(q - r)^{\frac{1}{2}}\left\{\tfrac{2^3}{3^3}(q - r)^{\frac{1}{8}} + \tfrac{2^4}{3^4}(q - r)^{\frac{1}{16}} + \tfrac{2^5}{3^5}(q - r)^{\frac{1}{32}} + \ldots\right\}
\end{aligned}
$$

is established.

Comparing this with the corresponding formula for Hamiltonian Numbers,

$$p = q^2 - \tfrac{2}{3}q\left(q^{\frac{1}{2}} + q^{\frac{1}{4}} + q^{\frac{1}{8}} + q^{\frac{1}{16}} + \ldots + q^{(\frac{1}{2})^i}\right) + \Xi q,$$

given at the beginning of the third section (at the top of p. 302, where the last term is incorrectly printed Ξ), it will be noticed that each of the two developments begins with an irregular portion consisting respectively of four and one terms, followed by a regular series. In the one case the regular portion is $\frac{11}{45}(q - r)^{\frac{1}{2}}$, multiplied by a series whose general term is $\frac{2^n}{3^n}(q - r)^{(\frac{1}{2})^n}$; in the other it consists of a series of terms of the form $q^{(\frac{1}{2})^n}$ multiplied by $-\frac{2}{3}q$.